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ScienceDirect
Stochastic Processes and their Applications 121 (2011) 1245–1265
stochastic processes and their applications
www.elsevier.com/locate/spa

Transition density estimates for jump Lévy processes

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Received 30 November 2010; received in revised form 1 March 2011; accepted 4 March 2011
Available online 12 March 2011

Abstract

Upper estimates of densities of convolution semigroups of probability measures are given under explicit assumptions on the corresponding Lévy measure and the Lévy–Khinchin exponent.

MSC: primary 60G51; 60E07; secondary 60J35; 47D03; 60J45

Keywords: Stable process; Layered stable process; Tempered stable process; Semigroup of measures; Transition density; Heat kernel

1. Introduction

We investigate a class of jump-type Lévy processes and the corresponding convolution semigroups of measures and give estimates of their transition densities. The primary examples of considered objects are the stable Lévy processes. Stable Lévy processes are important in theoretical probability, physics and finance, and their asymptotic properties are of interest for many authors. The estimates of their transition densities were obtained, e.g., in [2,28,16,17,13,14,36,4]. Our present study applies to the more general layered and tempered stable processes (for definitions, see Section 6) with marginal tails heavier than Gaussian but lighter than stable processes. The layered and tempered Lévy processes are known in the physical literature as *truncated Lévy flights* and were introduced in statistical physics to model turbulence [27,26,23]. Numerical analysis of stock price time series also indicate distributions of this type [37]. In particular they are used to model stochastic volatility [7,8]. Our main objective is to describe the radial decay of the transition density when the radial decay *and* some degree of smoothness of the

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2011.03.002

Lévy measure are given. In particular we obtain precise off-diagonal estimates of the transition density even if the Lévy measure is singular. We will now describe our results.

Let $d \in \{1, 2, \ldots\}$, $b \in \mathbb{R}^d$, and ν be a Lévy measure on $\mathbb{R}^d$, i.e.,

$$\int_{\mathbb{R}^d} \left(1 \wedge |y|^2\right) \nu(\mathrm{d}y) < \infty.$$

We consider the convolution semigroup of probability measures $\{P_t, t \geq 0\}$ with the Fourier transform $\mathcal{F}(P_t)(\xi) = \exp(-t\Phi(\xi))$, where

$$\Phi(\xi) = -\int \left(\mathrm{e}^{\mathrm{i}\xi \cdot y} - 1 - \mathrm{i}\xi \cdot y \mathbf{1}_{B(0,1)}(y)\right) \nu(\mathrm{d}y) - \mathrm{i}\xi \cdot b, \quad \xi \in \mathbb{R}^d. \tag{1}$$

In Section 2 we recall a procedure of the construction of the semigroup. The main result of the present paper is the following theorem.

Theorem 1. *Assume that ν is a Lévy measure such that*

$$\nu(A) \leq \int_{\mathbb{S}} \int_0^\infty \mathbf{1}_A(s\theta) s^{-1-\alpha} q(s) \phi(s) \,\mathrm{d}s \mu(\mathrm{d}\theta), \quad A \in \mathcal{B}(\mathbb{R}^d), \tag{2}$$

where $\alpha \in (0, 2)$, μ is a finite nonnegative measure on the unit sphere $\mathbb{S}$, and $q : [0, \infty) \to (0, \infty)$ is bounded nonincreasing function such that

$$q(s) \leq \kappa_1 q(2s), \quad s > 0, \tag{3}$$

for a positive constant κ_1, and $\phi : [0, \infty) \to (0, \infty)$ is a bounded nonincreasing function satisfying

$$\phi(s_1)\phi(s_2) \leq \kappa_2 \phi(s_1 + s_2), \quad s_1, s_2 \geq 0, \tag{4}$$

where $\kappa_2 > 0$.

Furthermore we assume that there exists $\beta \in [\alpha, 2]$ and finite $c_1, c_2 \geq 0$ such that

$$\int_0^r s^{1-\alpha} q(s) \frac{\phi(s)}{\phi(s/2)} \,\mathrm{d}s \leq c_1 r^{2-\beta}, \quad r > 1, \tag{5}$$

and

$$\Re(\Phi(\xi)) \geq c_2 \left(|\xi|^\alpha \wedge |\xi|^\beta\right), \quad \xi \in \mathbb{R}^d. \tag{6}$$

We finally assume that μ is a $\gamma - 1$-measure on $\mathbb{S}$ for some $\gamma \in [1, d]$, i.e.

$$\mu(\mathbb{S} \cap B(\theta, \rho)) \leq c\rho^{\gamma-1}, \quad \theta \in \mathbb{S},\ \rho > 0. \tag{7}$$

Then the measures P_t are absolutely continuous with respect to the Lebesgue measure and their densities p_t satisfy

$$p_t(x + tb_{t^{1/\alpha}}) \leq ct^{-d/\alpha} \min\left\{1, t^{1+\frac{\gamma}{\alpha}} |x|^{-\gamma-\alpha} q(|x|)\phi\left(|x|/4\right)\right\},$$

for $t \in (0, 1]$, $x \in \mathbb{R}^d$, and

$$p_t(x + tb_{t^{1/\beta}}) \leq c_1 t^{-d/\beta} \left(\min\left\{1, t^{1+\frac{\gamma}{\beta}} |x|^{-\gamma-\alpha} q(|x|)\phi\left(|x|/4\right)\right\} + \mathrm{e}^{-c_2 t^{-1/\beta}|x| \log(1+c_3 t^{-1/\beta}|x|)}\right), \tag{8}$$

for $t > 1$, $x \in \mathbb{R}^d$, *where*

$$b_r = \begin{cases} b - \int_{r<|y|<1} y\nu(\mathrm{d}y) & \textit{if } r \le 1, \\ b + \int_{1<|y|<r} y\nu(\mathrm{d}y) & \textit{if } r > 1. \end{cases} \tag{9}$$

In polar coordinates we have $\nu(\mathrm{d}s\mathrm{d}\theta) \le s^{-1-\alpha}q(s)\phi(s)\mathrm{d}s\mu(\mathrm{d}\theta)$. The *relative* radial decay of the upper bound is the same in each direction, but its *absolute* size is highly anisotropic, as permitted by the spectral measure μ. Condition (6) is our only lower bound for ν. In particular, if μ is nondegenerate, i.e., the support of μ is not contained in any proper linear subspace of $\mathbb{R}^d$, and for a constant $r_0 > 0$ we have

$$\nu(\mathrm{d}s\mathrm{d}\theta) \ge c\mathbf{1}_{(0,r_0]}(s)s^{-1-\alpha}\mathrm{d}s\mu(\mathrm{d}\theta),$$

then $\Re(\Phi(\xi)) \ge c\left(|\xi|^\alpha \wedge |\xi|^2\right)$. Furthermore, if for $\beta \in [\alpha, 2)$ we have

$$\nu(\mathrm{d}s\mathrm{d}\theta) \ge c\left[\mathbf{1}_{(0,r_0]}(s)s^{-1-\alpha} + \mathbf{1}_{(r_0,\infty)}(s)s^{-1-\beta}\right]\mathrm{d}s\mu(\mathrm{d}\theta),$$

then $\Re(\Phi(\xi)) \ge c\left(|\xi|^\alpha \wedge |\xi|^\beta\right)$. Condition (6) guarantees the existence (and smoothness) of the densities because it yields that the Fourier transform of P_t decays faster in infinity than every negative power of $|\xi|$.

We should also note that the *doubling property* (3) is equivalent to the following: there exist $c > 0$ and $\eta \ge 0$ such that

$$\frac{q(r)}{q(R)} \le c\left(\frac{r}{R}\right)^{-\eta}, \quad 0 < r \le R. \tag{10}$$

The typical examples of such functions q are $q(s) = (1+s)^{-a}$, or $q(s) = (\log(e+s))^a(1+s)^{-ma}$, for $a \ge 0$ and $m > 1$.

If a function ϕ satisfies (4) then $1/\phi$ is submultiplicative in the sense of Section 25 in [31] and locally bounded. By [31, Lemma 25.5] there exist c_1, c_2 such that $\phi(s) \ge c_1 e^{-c_2 s}$, $s \ge 0$. Functions which satisfy (4) are, e.g., $\exp(-ms^a)$, $1/\log^m(s+e)$ for every $m > 0$ and $0 < a \le 1$. A product of two functions which satisfy (4) also satisfies (4) (see [31, Proposition 25.4]). Conditions (3) and (4) are essential for estimates of convolutional powers of the bounded part of the Lévy measure in Section 4.

We note that (7) is equivalent to ν being a γ-measure near S : $\nu(B(\theta, r)) \le cr^\gamma$ for $\theta \in S, r < 1/2$.

We would like to mention recent results related to Theorem 1. The rotation invariant α-stable Lévy process has the Lévy measure $\nu(\mathrm{d}y) = c|y|^{-d-\alpha}$. The asymptotic behaviour of its densities is well known (see, e.g., [2]) and in this case we have $p_t(x) \approx \min(t^{-d/\alpha}, t|x|^{-d-\alpha})$. Pruitt and Taylor investigated in [28] multivariate stable densities in a more general setting. They obtained the upper bound $p_t(x) \le ct^{-d/\alpha}(1 + t^{-1/\alpha}|x|)^{-1-\alpha}$ by Fourier-analytic methods. We like to note that such decay is indeed observed if the spectral measure μ of the stable Lévy has an atom (for lower bounds see [16,17,36]). Using the perturbation formula Głowacki and Hebisch proved in [13,14] that if $\nu(\mathrm{d}x) = |x|^{-\alpha-d}g(x/|x|)\mathrm{d}x$ and g is bounded, then $p_t(x) \le c\min(t^{-d/\alpha}, t|x|^{-d-\alpha})$. When g is continuous on $\mathbb{S}$ we even have $\lim_{r\to\infty} r^{d+\alpha}p_1(r\theta) = cg(\theta), \theta \in \mathbb{S}$ and if $g(\theta) = 0$ then additionally $\lim_{r\to\infty} r^{d+2\alpha}p_1(r\theta) = c_\theta > 0$, which was proved by Dziubański in [12]. Zaigraev in [38] obtained further asymptotic expansions of the

α-stable density for sufficiently regular g. Bogdan and Jakubowski in [3] obtained estimates of heat kernels of the fractional Laplacian perturbed by gradient operators. Derivatives of stable densities have been considered in [25,35]. The existence and upper estimates of the densities for symmetric Lévy and Lévy-type processes were investigated also by Knopova and Schilling in [22,21].

More recent asymptotic results for stable Lévy processes are given in papers [36,4]. In particular if the Lévy measure is given in polar coordinates as $\nu(\mathrm{d}s\mathrm{d}\theta) = s^{-1-\alpha}\mathrm{d}s\mu(\mathrm{d}\theta)$ and μ is symmetric and satisfies (7) for some $\gamma \in [1, d]$ then we have $p_t(x) \leq c \min\{t^{-d/\alpha}, t^{1+(\gamma-d)/\alpha}|x|^{-\gamma-\alpha}\}$.

In the present paper we strengthen the results and the methods of [34], which was restricted to symmetric processes and $\phi \equiv 1$. We take this opportunity to notice that one of our estimates in [34] is incorrect. Namely, the upper bound of the density for large times t should contain $t^{1+\frac{\eta+\gamma-d}{\beta}}$ (where η is defined by (10)) instead of $t^{1+\frac{\gamma-d}{\beta}}$ according to the proof given in [34, Theorem 3] but the conclusion of the proof failed to summarize the estimates properly. Here we override the estimates with the more adequate (8).

We also want to mention estimates for the transition density of a class of Markov processes with jump intensities which are not necessarily translation invariant but dominated by the Lévy measure of the stable rotation invariant process given in [33] (see also [32]). It is an open and important problem to extend the present techniques to the Markov (not translation invariant) case. Existing results for Markov transition semigroups [10,11] are mostly based on assuming that the Lévy (jump) kernel of the Markov process has isotropic upper and lower bounds in small scales. In sharp contrast, our assumptions allow for a highly anisotropic behaviour of the Lévy measure ν. This is so because the spectral measure μ in (2) does not in general have a finite isotropic majorant, unless μ is (bounded by) the uniform measure on the unit sphere, which is a trivial case in our considerations.

The paper is organized as follows. In Section 2 we give preliminaries. Section 3 contains estimates of the transition densities for the Lévy measures with bounded support. In Section 4 we investigate asymptotics of the convolution exponents of finite measures. In Section 5 we prove Theorem 1 by combining the results of Sections 4 and 5. In Section 6 we give examples of processes satisfying the assumptions of Theorem 1.

2. Construction of the semigroup

For $x \in \mathbb{R}^d$ and $r > 0$ we let $|x| = \sqrt{\sum_{i=1}^d x_i^2}$ and $B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$. We denote $\mathbb{S} = \{x \in \mathbb{R}^d : |x| = 1\}$. All the sets, functions and measures considered in what follows will be Borel. For a measure λ on $\mathbb{R}^d$, $|\lambda|$ denotes its total mass. For a function f we let $\lambda(f) = \int f \mathrm{d}\lambda$, whenever the integral makes sense. When $|\lambda| < \infty$ and $n = 1, 2, \ldots$ we let λ^{n*} denote the n-fold convolution of λ with itself:

$$\lambda^{n*}(f) = \int f(x_1 + x_2 + \cdots + x_n)\lambda(\mathrm{d}x_1)\lambda(\mathrm{d}x_2)\ldots\lambda(\mathrm{d}x_n).$$

We also let $\lambda^{0*} = \delta_0$, the evaluation at 0, and

$$\mathrm{e}^{\lambda}(f) = \sum_{n=0}^{\infty} \frac{\lambda^{n*}(f)}{n!}.$$

Let ν be a Lévy measure on $\mathbb{R}^d$, i.e., $\int \left(|y|^2 \wedge 1\right) \nu(\mathrm{d}y) < \infty$. For all $0 \leq \rho < r \leq \infty$ we denote $\nu_{\rho,r}(\mathrm{d}y) = \mathbf{1}_{B(0,r)\setminus B(0,\rho)}(y)\nu(\mathrm{d}y)$.

We now construct the semigroup $\{P_t,\ t \geq 0\}$ corresponding to ν (for a more axiomatic introduction to convolution semigroups we refer the reader to [1,20]). We denote

$$a_{\rho,r} = -\int_{B(0,1)} y\nu_{\rho,r}(\mathrm{d}y), \quad 0 < \rho < r \leq \infty,$$

and consider the probability measures

$$P_t^{\rho,r} = \mathrm{e}^{t(\nu_{\rho,r} - |\nu_{\rho,r}|\delta_0)} * \delta_{ta_{\rho,r}}, \quad 0 < \rho < r \leq \infty,\ t \geq 0.$$

$P_t^{\rho,r}$ form a convolution semigroup:

$$P_t^{\rho,r} * P_s^{\rho,r} = P_{s+t}^{\rho,r}, \quad s, t \geq 0.$$

The Fourier transform of $P_t^{\rho,r}$ is

$$\mathcal{F}(P_t^{\rho,r})(\xi) = \int \mathrm{e}^{\mathrm{i}\xi\cdot y} P_t^{\rho,r}(\mathrm{d}y) = \exp\left(t \int \left(\mathrm{e}^{\mathrm{i}\xi\cdot y} - 1 - \mathrm{i}\xi \cdot y\mathbf{1}_{B(0,1)}(y)\right) \nu_{\rho,r}(\mathrm{d}y)\right),$$
$$\xi \in \mathbb{R}^d.$$

We have

$$\int |y|^2 P_t^{\rho,r}(\mathrm{d}y) = \sum_{k=1}^{d} \left[-\frac{\partial^2}{\partial\xi_k^2}\mathcal{F}(P_t^{\rho,r})(\xi)\right]_{\xi=0} = t\int |y|^2 \nu_{\rho,r}(\mathrm{d}y), \tag{11}$$

for all $0 < \rho < r \leq 1$. Furthermore $P_t^{\rho,1} = P_t^{r,1} * P_t^{\rho,r}$, and this yields

$$\begin{aligned} P_t^{\rho,1} - P_t^{r,1} &= P_t^{r,1} * P_t^{\rho,r} - P_t^{r,1} \\ &= P_t^{r,1} * \left(P_t^{\rho,r} - \delta_0\right). \end{aligned}$$

Let f be a continuous bounded function on $\mathbb{R}^d$. For every $\varepsilon > 0$ and $R > 0$ we can choose $\delta > 0$ such that $|f(x + y) - f(x)| < \varepsilon/3$ for all $x \in B(0, R)$, $|y| < \delta$, and by (11) and Chebyshev's inequality we get

$$\begin{aligned} \left|\int f(x+y)P_t^{\rho,r}(\mathrm{d}y) - f(x)\right| &= \left|\int (f(x+y) - f(x))\, P_t^{\rho,r}(\mathrm{d}y)\right| \\ &\leq \int |f(x+y) - f(x)|\, P_t^{\rho,r}(\mathrm{d}y) \\ &\leq \int_{|y|<\delta} \frac{\varepsilon}{3} P_t^{\rho,r}(\mathrm{d}y) + \int_{|y|\geq\delta} 2\|f\|_\infty P_t^{\rho,r}(\mathrm{d}y) \\ &\leq \frac{\varepsilon}{3} + 2\|f\|_\infty \frac{t\int |y|^2\nu_{0,r}(\mathrm{d}y)}{\delta^2}. \end{aligned}$$

Taking $R > 0$ such that $R^2 > \frac{6\|f\|_\infty t\int |y|^2\nu_{0,1}(\mathrm{d}y)}{\varepsilon}$ and $r > 0$ such that $\int |y|^2\nu_{0,r}(\mathrm{d}y) < \frac{\delta^2}{2t\|f\|_\infty}\frac{\varepsilon}{3}$ we obtain

$$\left|\left(P_t^{\rho,1} - P_t^{r,1}\right)(f)\right| \leq \int \left(\int |f(x+y) - f(x)|\, P_t^{\rho,r}(\mathrm{d}y)\right) P_t^{r,1}(\mathrm{d}x)$$

$$\leq \int_{B(0,R)} \left(\int |f(x+y) - f(x)| \, P_t^{\rho,r}(\mathrm{d}y) \right) P_t^{r,1}(\mathrm{d}x)$$
$$+ \int_{B(0,R)^c} 2\|f\|_\infty P_t^{r,1}(\mathrm{d}x)$$
$$\leq \frac{\varepsilon}{3} + 2\|f\|_\infty \frac{t \int |y|^2 \nu_{0,r}(\mathrm{d}y)}{\delta^2}$$
$$+ 2\|f\|_\infty \frac{t \int |y|^2 \nu_{0,1}(\mathrm{d}y)}{R^2} \leq \varepsilon.$$

It follows that the measures $P_t^{r,1}$ weakly converge to a probability measure $P_t^{0,1}$ as $r \to 0$. We let $P_t = P_t^{0,1} * P_t^{1,\infty} * \delta_{tb}$. $\{P_t,\ t \geq 0\}$ is also a convolution semigroup and $\mathcal{F}(P_t)(u) = \exp(-t\Phi(u))$, where

$$\Phi(\xi) = -\int \left(\mathrm{e}^{\mathrm{i}\xi\cdot y} - 1 - \mathrm{i}\xi \cdot y \mathbf{1}_{B(0,1)}(y) \right) \nu(\mathrm{d}y) - \mathrm{i}\xi \cdot b, \quad \xi \in \mathbb{R}^d.$$

We call ν the *Lévy measure* of the semigroup $\{P_t,\ t \geq 0\}$ [18,1]. The semigroup determines the stochastic Lévy process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ with transition probabilities $P(t, x, A) = P_t(A - x)$.

In what follows *constants* mean positive real numbers depending only on $d, \alpha, \beta, \mu, q, \phi, \gamma, \eta$. We write $f \approx g$ to indicate that there is a constant c such that $c^{-1} f \leq g \leq cf$.

3. Lévy measure with bounded support

We consider in this section a Lévy measure ν_0 such that $\mathrm{supp}\,\nu_0 \subset B(0, r)$ for some $r > 0$. Let π be a infinitely divisible distribution with generating triplet (see [31]) $(0, \nu_0, 0)$, i.e.

$$\mathcal{F}(\pi)(\xi) = \exp\left(\int (\mathrm{e}^{\mathrm{i}\xi\cdot y} - 1 - \mathrm{i}\xi \cdot y \mathbf{1}_{B(0,1)}(y)) \nu_0(\mathrm{d}y) \right), \quad \xi \in \mathbb{R}^d. \tag{12}$$

The Lévy measures with bounded support are discussed, e.g., in Section 26 of [31]. It follows from [31, Theorem 26.1] that $\pi(B(0,a)^c) = o(\mathrm{e}^{-\kappa a \log a})$ as $a \to \infty$ for all $\kappa \in (0, 1/r)$. In this section we will essentially use the methods of [31] but our computations have to be a little bit more precise. In particular, as a consequence of the estimate of tails given in Lemma 1 we obtain in Lemma 2 an estimate of a density of π. We note that sharp estimates for the transition densities of the truncated stable processes (with $\nu(\mathrm{d}x) = c|x|^{-d-\alpha} \mathbf{1}_{|x|<1} \mathrm{d}x$) were given in [9].

We let

$$\xi_0^j = \int_{|y|>1} y_j \nu_0(\mathrm{d}y), \quad j = 1, 2, \ldots, d,$$
$$\xi_0 = \max\{|\xi_0^1|, \ldots, |\xi_0^d|\},$$

and

$$M_j = \int y_j^2 \nu_0(\mathrm{d}y), \quad j = 1, 2, \ldots, d,$$
$$M = \max\{M_1, \ldots, M_d\}.$$

Lemma 1. *For every $a \geq 2\sqrt{d}\,(\xi_0 + M/e)$ we have*

$$\pi(B(0,a)^c) \leq 2d \exp\left(-\frac{a}{2\sqrt{d}(r+1)} \log \frac{a}{2\sqrt{d}M} \right). \tag{13}$$

Proof. It follows from [31, Proposition 11.10] that the measure

$$\pi_j(A) = \pi(\{y \in \mathbb{R}^d : y_j \in A\}), \quad A \subset \mathbb{R},$$

is infinitely divisible distribution on $\mathbb{R}$ with the generating triplet $(0, \nu_0^j, \gamma_j)$, where

$$\nu_0^j(A) = \nu_0(\{y \in \mathbb{R}^d : y_j \in A\}), \quad A \subset \mathbb{R},$$

and

$$\gamma_j = \int y_j \left(\mathbf{1}_{[-1,1]}(y_j) - \mathbf{1}_{B(0,1)}(y)\right) \nu_0(\mathrm{d}y).$$

Therefore by [31, Lemma 26.4] we obtain

$$\pi(\{y \in \mathbb{R}^d : y_j > b\}) \le \exp\left(-\int_{\xi_0^j}^b \theta_j(\xi)\mathrm{d}\xi\right),$$

for every $b \in (\xi_0^j, \infty)$, where $\theta_j(\xi)$ is the inverse function of

$$\psi_j(u) = \int y_j \left(\mathrm{e}^{u y_j} - \mathbf{1}_{B(0,1)}(y)\right) \nu_0(\mathrm{d}y), \quad u \in \mathbb{R}.$$

We note that ψ_j is continuous and increasing,

$$\psi_j(0) = \xi_0^j,$$

and $\lim_{u\to\infty} \psi_j(u) < \infty$ if and only if

$$\nu_0\left(\{y : y_j > 0\}\right) = 0, \quad \text{and} \quad \int_{-r<y_j<0} |y_j| \nu_0(\mathrm{d}y) < \infty.$$

In this case $\mathrm{supp}\pi_j \subset (-\infty, 0)$ (see [31, Corollary 24.8]). If

$$\lim_{u\to\infty} \psi_j(u) = \infty,$$

then θ_j is well defined on (ξ_0^j, ∞). For every $\xi \in (\xi_0^j, \infty)$ we have

$$\begin{aligned}
\xi = \psi_j(\theta_j(\xi)) &= \int y_j \left(\mathrm{e}^{\theta_j(\xi) y_j} - \mathbf{1}_{B(0,1)}(y)\right) \nu_0(\mathrm{d}y) \\
&= \int y_j \left(\mathrm{e}^{\theta_j(\xi) y_j} - 1\right) \nu_0(\mathrm{d}y) + \xi_0^j \\
&\le \mathrm{e}^{r\theta_j(\xi)} \theta_j(\xi) \int y_j^2 \, \nu_0(\mathrm{d}y) + \xi_0^j \le \frac{1}{e} \mathrm{e}^{\theta_j(\xi)(r+1)} M_j + \xi_0^j.
\end{aligned}$$

We obtain

$$\log(\xi - \xi_0^j) \le \theta_j(\xi)(r+1) - 1 + \log M_j,$$

which yields

$$\theta_j(\xi) \ge \frac{1}{r+1} \log\left(e \frac{\xi - \xi_0^j}{M_j}\right).$$

It follows that

$$\pi(\{y\in\mathbb{R}^d : y_j > b\}) \le \exp\left[-\int_{\xi_0^j}^{b} \frac{1}{r+1}\log\left(e\frac{\xi-\xi_0^j}{M_j}\right)\mathrm{d}\xi\right]$$
$$= \exp\left[-\frac{b-\xi_0^j}{r+1}\log\frac{b-\xi_0^j}{M_j}\right], \quad b > 0\vee \xi_0^j.$$

Similarly,

$$\pi(\{y\in\mathbb{R}^d : y_j < -b\}) = \pi(\{y\in\mathbb{R}^d : -y_j > b\})$$
$$\le \exp\left[-\int_{-\xi_0^j}^{b} \frac{1}{r+1}\log\left(e\frac{\xi+\xi_0^j}{M_j}\right)\mathrm{d}\xi\right]$$
$$= \exp\left[-\frac{b+\xi_0^j}{r+1}\log\frac{b+\xi_0^j}{M_j}\right], \quad b > 0\vee (-\xi_0^j).$$

We get

$$\pi\left(B(0,a)^c\right) \le \sum_{j=1}^{d}\pi\left(\left\{y\in\mathbb{R}^d : |y_j| > \frac{a}{\sqrt{d}}\right\}\right)$$
$$\le \sum_{j=1}^{d}\exp\left[-\frac{\frac{a}{\sqrt{d}}-\xi_0^j}{r+1}\log\frac{\frac{a}{\sqrt{d}}-\xi_0^j}{M_j}\right]$$
$$+\sum_{j=1}^{d}\exp\left[-\frac{\frac{a}{\sqrt{d}}+\xi_0^j}{r+1}\log\frac{\frac{a}{\sqrt{d}}+\xi_0^j}{M_j}\right], \quad a > \sqrt{d}\xi_0.$$

For $a > 2\sqrt{d}\xi_0$ we have

$$\frac{a}{\sqrt{d}} \pm \xi_0^j \ge \frac{a}{\sqrt{d}} - \xi_0 \ge \frac{a}{\sqrt{d}} - \frac{a}{2\sqrt{d}} = \frac{a}{2\sqrt{d}}.$$

The function $f(s) = s\log s$ is increasing on $[1/e,\infty)$, which yields

$$\exp\left[-\frac{\frac{a}{\sqrt{d}}\pm\xi_0^j}{r+1}\log\frac{\frac{a}{\sqrt{d}}\pm\xi_0^j}{M_j}\right] \le \exp\left(-\frac{a}{2\sqrt{d}(r+1)}\log\frac{a}{2\sqrt{d}M}\right),$$

provided $a \ge 2\sqrt{d}\left(\xi_0 + \frac{M}{e}\right)$, and the lemma follows. $\square$

Lemma 2. *Assume that π has a density $p(x)$ and let m_0, m_1 be such that*

$$\sup_{x\in\mathbb{R}^d} p(x) \le m_0, \qquad \max_{|\beta|=1}\sup_{x\in\mathbb{R}^d}|\partial^\beta p(x)| \le m_1. \tag{14}$$

Then there exist constants c_1, c_2, c_3 such that

$$p(x) \le c_1 m_1^{d/(d+1)}\exp\left(-\frac{c_2|x|}{r+1}\log\frac{c_3|x|}{M}\right), \tag{15}$$

for $|x| > \max\left\{4\sqrt{d}\left(\xi_0 + M/e\right), \frac{m_0}{m_1\sqrt{d}}\right\}$.

Proof. It follows from Lagrange's theorem that for every pair $x, y \in \mathbb{R}^d$ there exists $\theta \in [0, 1]$ such that

$$p(y) = p(x) + \nabla p(x + \theta(y - x)) \cdot (y - x),$$

and if $|y - x| \leq \frac{p(x)}{2m_1\sqrt{d}}$ then we obtain

$$p(y) \geq p(x) - m_1\sqrt{d}|y - x| \geq \frac{1}{2}p(x). \tag{16}$$

Let $|x| > \max\left\{4\sqrt{d}\,(\xi_0 + M/e)\,, \frac{m_0}{m_1\sqrt{d}}\right\}$. By Lemma 1 and (16) we get

$$\begin{aligned} 2d\exp\left(-\frac{|x|/2}{2\sqrt{d}(r+1)}\log\frac{|x|/2}{2\sqrt{d}M}\right) &\geq \pi\left(B(0, |x|/2)^c\right) \\ &\geq \pi\left(B\left(x, \frac{p(x)}{2m_1\sqrt{d}}\right)\right) \\ &= \int_{B\left(x, \frac{p(x)}{2m_1\sqrt{d}}\right)} p(y)\,\mathrm{d}y \\ &\geq \frac{1}{2}p(x)c\left(\frac{p(x)}{m_1\sqrt{d}}\right)^d, \end{aligned}$$

and we obtain

$$p(x) \leq c_1 m_1^{\frac{\mathrm{d}}{\mathrm{d}+1}} \exp\left(-\frac{c_2|x|}{r+1}\log\frac{c_3|x|}{M}\right). \quad \square$$

4. Large jumps

We assume in this section that ν is a Lévy measure which satisfies (2)–(4) and (7), i.e.

$$\nu(A) \leq \int_{\mathbb{S}}\int_0^\infty \mathbf{1}_A(s\theta)s^{-1-\alpha}q(s)\phi(s)\,\mathrm{d}s\mu(\mathrm{d}\theta), \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where

$$q(s) \leq \kappa_1 q(2s), \quad s \geq 0,$$

for some constant $\kappa_1 > 0$,

$$\phi(s_1)\phi(s_2) \leq \kappa_2\phi(s_1 + s_2), \quad s_1, s_2 \geq 0,$$

for some constant $\kappa_2 > 0$ and

$$\mu(\mathbb{S} \cap B(\theta, \rho)) \leq c\rho^{\gamma-1}, \quad \theta \in \mathbb{S},\ \rho > 0,$$

for some constant $\gamma \in [1, d]$.

For $r > 0$ we denote $\bar{\nu}_r(\mathrm{d}y) = \nu_{r,\infty}(\mathrm{d}y) = \mathbf{1}_{B(0,r)^c}(y)\nu(\mathrm{d}y)$.

For a set $A \subset \mathbb{R}^d$ we denote $\delta(A) = \mathrm{dist}(0, A) = \inf\{|y| : y \in A\}$ and $\mathrm{diam}(A) = \sup\{|y - x| : x, y \in A\}$.

For every set $A \subset \mathbb{R}^d$ such that $\delta(A) > 0$ and every $x_0 \in A$ we have

$$\begin{aligned}
\bar{\nu}_r(A) &\le \int_{\mathbb{S}} \int_{\delta(A)}^{\infty} \mathbf{1}_{B(x_0,\mathrm{diam}(A))}(s\theta) s^{-1-\alpha} q(s)\phi(s)\, \mathrm{d}s\, \mu(\mathrm{d}\theta) \\
&\le \mu\left(B\left(\frac{x_0}{|x_0|}, \frac{2\mathrm{diam}(A)}{|x_0|}\right)\right) 2\mathrm{diam}(A)(\delta(A))^{-1-\alpha} q(\delta(A))\phi(\delta(A)) \\
&\le c_1 \left(\frac{2\mathrm{diam}(A)}{|x_0|}\right)^{\gamma-1} (\delta(A))^{-1-\alpha} q(\delta(A))\phi(\delta(A))\mathrm{diam}(A) \\
&\le c_2(\delta(A))^{-\gamma-\alpha} q(\delta(A))\phi(\delta(A))\, (\mathrm{diam}(A))^{\gamma}\,. \qquad (17)
\end{aligned}$$

We will extend the estimate to convolutional powers of $\bar{\nu}_r$. We let

$$\psi(r) = |\mu|\phi(0) \int_r^{\infty} s^{-1-\alpha} q(s) \frac{\phi(s)}{\phi(s/2)}\, \mathrm{d}s, \quad r \in (0,\infty),$$

and we note that

$$|\bar{\nu}_r| \le |\mu| \int_r^{\infty} s^{-1-\alpha} q(s)\phi(s)\, \mathrm{d}s \le \psi(r), \quad r \in (0,\infty).$$

Lemma 3. *There exists a constant c such that*

$$\bar{\nu}_r^{n*}(A) \le c^n\, (\psi(r))^{n-1}\, (\delta(A))^{-\gamma-\alpha}\, q(\delta(A))\phi(\delta(A)/2)\, (\mathrm{diam}(A))^{\gamma} \qquad (18)$$

for every set A such that $\delta(A) > 0$ *and all* $n \in \mathbb{N}, r > 0$.

Proof. We use induction. Let (18) hold for some $n \in \mathbb{N}$ and constant c_0 and A be a set such that $\delta(A) > 0$. For $y \in \mathbb{R}^d$ we denote $D_y = \{z \in \mathbb{R}^d : |z| > \frac{1}{2}|z+y|\} = B\left(\frac{1}{3}y, \frac{2}{3}|y|\right)^c$. We have

$$\begin{aligned}
\bar{\nu}_r^{(n+1)*}(A) &= \int \bar{\nu}_r(A-y)\bar{\nu}_r^{n*}(\mathrm{d}y) \\
&= \int \bar{\nu}_r\left((A-y)\cap D_y\right)\bar{\nu}_r^{n*}(\mathrm{d}y) + \int \bar{\nu}_r\left((A-y)\cap D_y^c\right)\bar{\nu}_r^{n*}(\mathrm{d}y) \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

We note that for $z \in (A-y)\cap D_y$ we have $z+y \in A$ and $|z| > \frac{1}{2}|z+y|$, therefore $|z| > \frac{1}{2}\delta(A)$ and $\delta((A-y)\cap D_y) > \frac{1}{2}\delta(A)$. Furthermore, $\mathrm{diam}((A-y)\cap D_y) \le \mathrm{diam}(A)$ and using (17) and (10) we obtain

$$\begin{aligned}
\mathrm{I} &\le c_1 2^{\gamma+\alpha+\eta}(\delta(A))^{-\gamma-\alpha} q(\delta(A))\phi\,(\delta(A)/2)\,(\mathrm{diam}(A))^{\gamma}\, |\bar{\nu}_r^{n*}| \\
&\le c_1 2^{\gamma+\alpha+\eta}\, (\psi(r))^n\, (\delta(A))^{-\gamma-\alpha} q(\delta(A))\phi\,(\delta(A)/2)\,(\mathrm{diam}(A))^{\gamma}\,.
\end{aligned}$$

We have

$$\begin{aligned}
\mathrm{II} &= \iint \mathbf{1}_{A-y}(z)\mathbf{1}_{D_y^c}(z)\bar{\nu}_r(\mathrm{d}z)\bar{\nu}_r^{n*}(\mathrm{d}y) \\
&= \iint \mathbf{1}_{A-z}(y)\mathbf{1}_{B(-z,2|z|)^c}(y)\bar{\nu}_r^{n*}(\mathrm{d}y)\bar{\nu}_r(\mathrm{d}z) \\
&= \int \bar{\nu}_r^{n*}\left((A-z)\cap B(-z,2|z|)^c\right)\bar{\nu}_r(\mathrm{d}z).
\end{aligned}$$

Let $y \in V_z := (A - z) \cap B(-z, 2|z|)^c$. We then have $y + z \in A$, so $|y + z| \geq \delta(A)$, and $|y + z| \geq 2|z|$. Furthermore $|y| \geq |y + z| - |z|$ and this yields

$$\delta(V_z) \geq \inf_{y \in V_z} |y + z| - |z| \geq (\delta(A) \vee 2|z|) - |z| \geq \frac{1}{2}\delta(A),$$

and by (3), (4), (10) and induction we get

$$\begin{aligned}
\mathrm{II} &\leq \int c_0^n \,(\psi(r))^{n-1} \left(\frac{1}{2}\delta(A)\right)^{-\gamma-\alpha} q\left(\frac{1}{2}\delta(A)\right) \phi \\
&\quad \times \left(\frac{1}{2}\,(\delta(A) \vee 2|z|) - \frac{1}{2}|z|\right) (\mathrm{diam}(A))^{\gamma}\, \bar{\nu}_r(\mathrm{d}z) \\
&\leq c_0^n c_2 \,(\psi(r))^{n-1}\, 2^{\gamma+\alpha+\eta}\, (\delta(A))^{-\gamma-\alpha}\, q\,(\delta(A))\, (\mathrm{diam}(A))^{\gamma}\, \kappa_2 \int \frac{\phi\,(\delta(A)/2)}{\phi(|z|/2)} \bar{\nu}_r(\mathrm{d}z) \\
&\leq c_0^n c_2 \,(\psi(r))^{n-1}\, 2^{\gamma+\alpha+\eta}\, (\delta(A))^{-\gamma-\alpha}\, q\,(\delta(A))\, \phi\,(\delta(A)/2)\, (\mathrm{diam}(A))^{\gamma} \\
&\quad \times \kappa_2 |\mu| \int_r^{\infty} \frac{1}{\phi(s/2)} s^{-1-\alpha} q(s)\phi(s)\,\mathrm{d}s \\
&= c_0^n \kappa_2 c_2 2^{\gamma+\alpha+\eta}\, (\psi(r))^{n-1}\, (\delta(A))^{-\gamma-\alpha}\, q\,(\delta(A))\, \phi\,(\delta(A)/2)\, (\mathrm{diam}(A))^{\gamma}\, \frac{\psi(r)}{\phi(0)}.
\end{aligned}$$

The lemma follows by taking $c_0 \geq \max\{1, 2^{\gamma+\alpha+\eta}(c_1 + \kappa_2 c_2/\phi(0))\}$. $\square$

Corollary 4. *There exists a constant c such that for every $x \neq 0$, $n \in \mathbb{N}$ and $\rho < \frac{1}{2}|x|$ we have*

$$\bar{\nu}_r^{n*}(B(x, \rho)) \leq c^n \,(\psi(r))^{n-1}\, |x|^{-\gamma-\alpha} q(|x|)\phi\,(|x|/4)\, \rho^{\gamma}. \tag{19}$$

We consider the probability measures $\{\bar{P}_t^r,\ t \geq 0\}$ such that

$$\mathcal{F}(\bar{P}_t^r)(\xi) = \exp\left(t \int (\mathrm{e}^{\mathrm{i}\xi \cdot y} - 1)\bar{\nu}_r(\mathrm{d}y)\right), \quad \xi \in \mathbb{R}^d. \tag{20}$$

Note that

$$\begin{aligned}
\bar{P}_t^r &= P_t^{r,\infty} * \delta_{-ta_{r,1}} = \exp(t\,(\bar{\nu}_r - |\bar{\nu}_r|\delta_0)) = \sum_{n=0}^{\infty} \frac{t^n\,(\bar{\nu}_r - |\bar{\nu}_r|\delta_0)^{n*}}{n!} \\
&= \mathrm{e}^{-t|\bar{\nu}_r|} \sum_{n=0}^{\infty} \frac{t^n \bar{\nu}_r^{n*}}{n!}, \quad t \geq 0.
\end{aligned} \tag{21}$$

We obtain

Corollary 5. *There exists a constant c such that*

$$\bar{P}_t^r\,(B(x, \rho)) \leq ct\mathrm{e}^{t(c\psi(r) - |\bar{\nu}_r|)} |x|^{-\gamma-\alpha} q(|x|)\phi\,(|x|/4)\, \rho^{\gamma}, \tag{22}$$

for every $x \in \mathbb{R}^d \setminus \{0\}$, and $\rho < \frac{1}{2}|x|$.

5. Proof of Theorem 1

For the proof of Theorem 1 we split the Lévy measure ν into two parts: $\nu(\mathrm{d}y) = \mathbf{1}_{B(0,r)}(y)\nu(\mathrm{d}y) + \mathbf{1}_{B(0,r)^c}(y)\nu(\mathrm{d}y)$. For the semigroup generated by the first part we can use the results of Section 3 and for the second *large jump part* we use the estimates from Section 4. Then we can combine the results using the convolution.

We begin with the following technical lemma.

Lemma 6. *Let $a > v \geq 0$. If $f : (0, \infty) \mapsto \mathbb{R}$ is a nonnegative function such that*

$$\int_0^r s^a f(s)\, \mathrm{d}s \leq c_1 r^v, \quad r \geq r_0, \tag{23}$$

for some constants $c_1, r_0 > 0$, then

$$\int_r^\infty f(s)\, \mathrm{d}s \leq \frac{c_1 a}{a-v} r^{v-a}, \quad r \geq r_0.$$

Proof. For $r > r_0$ by (23) we have

$$\int_r^\infty \frac{1}{t^{a+1}} \left(\int_0^t s^a f(s)\, \mathrm{d}s \right) \mathrm{d}t \leq \int_r^\infty c_1 t^{v-a-1} \mathrm{d}t = \frac{c_1}{a-v} r^{v-a}.$$

Furthermore, changing the order of integration we obtain

$$\begin{aligned} \int_r^\infty \frac{1}{t^{a+1}} \left(\int_0^t s^a f(s)\, \mathrm{d}s \right) \mathrm{d}t &= \int_0^r s^a f(s) \int_r^\infty \frac{1}{t^{a+1}}\, \mathrm{d}t\, \mathrm{d}s \\ &\quad + \int_r^\infty s^a f(s) \int_s^\infty \frac{1}{t^{a+1}}\, \mathrm{d}t\, \mathrm{d}s \\ &= \frac{1}{ar^a} \int_0^r s^a f(s)\, \mathrm{d}s + \frac{1}{a} \int_r^\infty f(s)\, \mathrm{d}s \\ &\geq \frac{1}{a} \int_r^\infty f(s)\, \mathrm{d}s, \end{aligned}$$

and the lemma follows. □

For $r > 0$ we denote $\tilde{\nu}_r(\mathrm{d}y) = \nu_{0,r}(\mathrm{d}y) = \mathbf{1}_{B(0,r)}(y)\nu(\mathrm{d}y)$. We consider the semigroup of measures $\{\tilde{P}_t^r,\ t \geq 0\}$ such that

$$\mathcal{F}(\tilde{P}_t^r)(\xi) = \exp\left(t \int \left(\mathrm{e}^{\mathrm{i}\xi\cdot y} - 1 - \mathrm{i}\xi \cdot y \right) \tilde{\nu}_r(\mathrm{d}y) \right), \quad \xi \in \mathbb{R}^d.$$

By (6) we get

$$\begin{aligned} |\mathcal{F}(\tilde{P}_t^r)(\xi)| &= \exp\left(-t \int_{|y|<r} (1 - \cos(y \cdot \xi))\nu(\mathrm{d}y) \right) \\ &= \exp\left(-t \left(\Re(\Phi(\xi)) - \int_{|y|\geq r} (1 - \cos(y \cdot \xi))\nu(\mathrm{d}y) \right) \right) \\ &\leq \exp(-t\Re(\Phi(\xi))) \exp(2t\nu(B(0,r)^c)) \\ &\leq \exp\left(-ct \left(|\xi|^\alpha \wedge |\xi|^\beta \right) \right) \exp\left(2t\nu(B(0,r)^c) \right), \quad \xi \in \mathbb{R}^d. \end{aligned} \tag{24}$$

It follows that for every $r > 0$ and $t > 0$ the measure $\tilde{P}_t^r$ is absolutely continuous with respect to the Lebesgue measure with smooth density, say, $\tilde{p}_t^r$.

Let $h(t) = t^{1/\alpha} \wedge t^{1/\beta}$. Informally, $h(t)$ gives a correct (time-dependent) threshold for truncating the Lévy measure of convolutional exponents considered as approximations of the original transition probability p_t. We will often use $\tilde{P}_t^r$ and $\tilde{p}_t^r$ with $r = h(t)$ and for simplification we will denote

$$\tilde{P}_t = \tilde{P}_t^{h(t)} \quad \text{and} \quad \tilde{p}_t = \tilde{p}_t^{h(t)}.$$

Lemma 7. *If (2), (5) and (6) hold then*

$$\tilde{p}_t(x) \le c_1[h(t)]^{-d} \exp\left(-c_2 \frac{|x|}{h(t)} \log\left(1 + c_3 \frac{|x|}{h(t)}\right)\right), \tag{25}$$

for every $x \in \mathbb{R}^d \setminus \{0\}$, $t > 0$.

Proof. Let $g_t(y) = (h(t))^d \tilde{p}_t(h(t)y)$. It follows from (2) that for $r < 1$ we have $\nu(B(0,r)^c) \le (|\mu| q(0)\phi(0)/\alpha)\, r^{-\alpha}$, and from (5) and Lemma 6 with $a = 2$ and $v = 2 - \beta$ for $r > 1$ we get

$$\begin{aligned}
\nu(B(0,r)^c) &\le |\mu| \int_r^\infty s^{-1-\alpha} q(s)\phi(s)\, \mathrm{d}s \\
&\le \phi(0)|\mu| \int_r^\infty s^{-1-\alpha} q(s) \frac{\phi(s)}{\phi(s/2)}\, \mathrm{d}s \le c_1 r^{-\beta}.
\end{aligned}$$

From (24) for every $j \in \{1, \dots, d\}$ we obtain

$$\begin{aligned}
\left|\frac{\partial g_t}{\partial y_j}(y)\right| &= (h(t))^{d+1} \left|(2\pi)^{-d} \int_{\mathbb{R}^d} (-i)\xi_j \mathrm{e}^{-\mathrm{i}h(t)y\cdot\xi} \mathcal{F}(\tilde{p}_t)(\xi) \mathrm{d}\xi\right| \\
&\le c_2 (h(t))^{d+1} \left(\int_{|\xi|\le 1} |\xi_j| \mathrm{e}^{-c_3 t|\xi|^\beta}\, \mathrm{d}\xi + \int_{|\xi|>1} |\xi_j| \mathrm{e}^{-c_3 t|\xi|^\alpha}\, \mathrm{d}\xi\right) \\
&= c_2 (h(t))^{d+1} \left(t^{\frac{-1-d}{\beta}} \int_{|u|\le t^{1/\beta}} |u_j| \mathrm{e}^{-c_3|u|^\beta}\, \mathrm{d}u + t^{\frac{-1-d}{\alpha}} \int_{|u|>t^{1/\alpha}} |u_j| \mathrm{e}^{-c_3|u|^\alpha}\, \mathrm{d}u\right) \\
&\le c_4 (h(t))^{d+1} \left(t^{\frac{-1-d}{\beta}} + t^{\frac{-1-d}{\alpha}}\right) \\
&\le 2c_4.
\end{aligned}$$

Similarly we get $g_t(y) \le c_5$ for all $y \in \mathbb{R}^d$, $t > 0$.

We consider the infinitely divisible distribution $\pi_t(\mathrm{d}y) = g_t(y)\, \mathrm{d}y$. We note that

$$\begin{aligned}
\mathcal{F}(\pi_t)(\xi) &= \exp\left(t \int \left(\mathrm{e}^{\mathrm{i}\xi(h(t))^{-1}\cdot y} - 1 - \mathrm{i}\xi(h(t))^{-1} \cdot y \mathbf{1}_{B(0,h(t))}(y)\right) \tilde{\nu}_{h(t)}(\mathrm{d}y)\right) \\
&= \exp\left(\int \left(\mathrm{e}^{\mathrm{i}\xi\cdot y} - 1 - \mathrm{i}\xi \cdot y \mathbf{1}_{B(0,1)}(y)\right) \lambda_t(\mathrm{d}y)\right), \quad \xi \in \mathbb{R}^d,
\end{aligned}$$

where $\lambda_t(A) = t\tilde{\nu}_{h(t)}(h(t)A)$ is the Lévy measure of π_t. By (5) we have

$$\begin{aligned}
\int |y|^2 \lambda_t(\mathrm{d}y) &= t \int (|y|/h(t))^2 \tilde{\nu}_{h(t)}(\mathrm{d}y) \\
&\le |\mu| t (h(t))^{-2} \int_0^{h(t)} s^{1-\alpha} q(s)\phi(s)\, \mathrm{d}s \\
&\le c_6,
\end{aligned}$$

and

$$\int_{|y|>1} y_j \lambda_t(\mathrm{d}y) = t(h(t))^{-1} \int_{B(0,h(t))^c} y_j \tilde{\nu}_{h(t)}(\mathrm{d}y) = 0.$$

It follows from Lemma 2 that

$$g_t(y) \le c_7 \exp\left(-c_8|y| \log\left(c_9|y|\right)\right) \le c_{10} \exp\left(-c_8|y| \log\left(1 + c_9|y|\right)\right),$$

for $y \in \mathbb{R}^d \setminus \{0\}$, and this yields

$$\tilde{p}_t(x) \le c_{10}(h(t))^{-d} \exp\left(-\frac{c_8|x|}{h(t)} \log\left(1 + \frac{c_9|x|}{h(t)}\right)\right),$$

for $x \in \mathbb{R}^d \setminus \{0\}$, $t > 0$. $\square$

We will prove now the on-diagonal estimates of the density p_t. We observe that Nash-type inequalities [6] can be used to prove such estimates in the case of (symmetric) Markov semigroups. The present situation of convolutional semigroups is, however, simpler, and we can use Fourier transform instead.

We note that the existence (and smoothness) of the densities follows from (6) because (6) yields that the Fourier transform of P_t decays faster in infinity than every negative power of $|\xi|$.

Lemma 8. *If (6) holds then there exists a constant c such that*

$$p_t(x) \le c[h(t)]^{-d},$$

for every $x \in \mathbb{R}^d$ *and* $t > 0$.

Proof. By (6) we have

$$\begin{aligned}
p_t(x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\cdot\xi} \mathrm{e}^{-t\Phi(\xi)}\, \mathrm{d}\xi \\
&\le (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathrm{e}^{-t\Phi(\xi)}|\, \mathrm{d}\xi = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{-t\Re(\Phi(\xi))}\, \mathrm{d}\xi \\
&\le (2\pi)^{-d} \left(\int_{|\xi|\le 1} \mathrm{e}^{-c_1 t|\xi|^\beta}\, \mathrm{d}\xi + \int_{|\xi|\ge 1} \mathrm{e}^{-c_1 t|\xi|^\alpha}\, \mathrm{d}\xi \right) \\
&= (2\pi)^{-d} \left(t^{-d/\beta} \int_{|u|\le t^{1/\beta}} \mathrm{e}^{-c_1|u|^\beta}\, \mathrm{d}u + t^{-d/\alpha} \int_{|u|\ge t^{1/\alpha}} \mathrm{e}^{-c_1|u|^\alpha}\, \mathrm{d}u \right) \\
&\le c_2 \left(t^{-d/\beta} + t^{-d/\alpha} \right) \\
&\le 2c_2[h(t)]^{-d}, \quad x \in \mathbb{R}^d,\ t > 0. \quad \square
\end{aligned}$$

Proof of Theorem 1. We have

$$P_t = \tilde{P}_t^r * \bar{P}_t^r * \delta_{tb_r}, \quad t \ge 0,$$

where $\bar{P}_t^r$ is defined by (20) and b_r by (9). Of course

$$p_t = \tilde{p}_t^r * \bar{P}_t^r * \delta_{tb_r}, \quad t > 0. \tag{26}$$

We will denote

$$\bar{P}_t = \bar{P}_t^{h(t)}.$$

By (5) and Lemma 6 we have

$$\psi(h(t)) = |\mu|\phi(0)\int_{h(t)}^{\infty} s^{-1-\alpha}q(s)\frac{\phi(s)}{\phi(s/2)}\,\mathrm{d}s$$
$$\leq \frac{c}{t}, \quad t > 0.$$

It follows from Corollary 5 that

$$\bar{P}_t(B(x,\rho)) \leq ct|x|^{-\gamma-\alpha}q(|x|)\phi\,(|x|/4)\,\rho^{\gamma}, \tag{27}$$

for $\rho \leq \frac{1}{2}|x|$ and $t > 0$.

We denote

$$g(s) = \mathrm{e}^{-c_2 s\log(1+c_3 s)}, \quad s \geq 0,$$

where constants c_2, c_3 are given by (25). We note that g is decreasing, continuous on $[0,\infty)$ and $g(s) \leq cs^{-2\gamma}$, for some $c > 0$, which yields that the inverse function $g^{-1} : (0,1] \to [0,\infty)$ exists, is decreasing, and $g^{-1}(s) \leq (c/s)^{1/(2\gamma)}$. In particular

$$\int_0^1 \left(g^{-1}(s)\right)^{\gamma}\,\mathrm{d}s < \infty.$$

Using Lemma 7 and (27) we obtain

$$\begin{aligned}
\tilde{p}_t * \bar{P}_t(x) &= \int \tilde{p}_t(x-y)\bar{P}_t(\mathrm{d}y) \\
&\leq \int c[h(t)]^{-d}g(|x-y|/h(t))\bar{P}_t(\mathrm{d}y) \\
&= c[h(t)]^{-d}\int\int_0^{g(|x-y|/h(t))}\mathrm{d}s\,\bar{P}_t(\mathrm{d}y) \\
&= c[h(t)]^{-d}\int_0^1\int \mathbf{1}_{\{y\in\mathbb{R}^d:g(|x-y|/h(t))>s\}}\bar{P}_t(\mathrm{d}y)\mathrm{d}s \\
&= c[h(t)]^{-d}\int_0^1 \bar{P}_t\left(B(x,h(t)g^{-1}(s))\right)\mathrm{d}s \\
&\leq c[h(t)]^{-d}\left(\int_{g\left(\frac{|x|}{2h(t)}\right)}^1 t|x|^{-\gamma-\alpha}q(|x|)\phi\,(|x|/4)\left(h(t)g^{-1}(s)\right)^{\gamma}\,\mathrm{d}s\right. \\
&\quad \left. + \int_0^{g\left(\frac{|x|}{2h(t)}\right)}\mathrm{d}s\right) \\
&\leq c[h(t)]^{-d}\left(t[h(t)]^{\gamma}|x|^{-\gamma-\alpha}q(|x|)\phi\,(|x|/4)\int_0^1\left(g^{-1}(s)\right)^{\gamma}\,\mathrm{d}s\right. \\
&\quad \left. + g\left(\frac{|x|}{2h(t)}\right)\right) \\
&= c[h(t)]^{-d}\left(t[h(t)]^{\gamma}|x|^{-\gamma-\alpha}q(|x|)\phi\,(|x|/4) + g\left(\frac{|x|}{2h(t)}\right)\right).
\end{aligned}$$

This and Lemma 8 yield

$$
\begin{aligned}
p_t(x+tb_{h(t)}) &= \int \tilde{p}_t * \bar{P}_t(x+tb_{h(t)}-y)\delta_{tb_{h(t)}}(\mathrm{d}y) \\
&= \tilde{p}_t * \bar{P}_t(x) \\
&\le c[h(t)]^{-d} \min\left\{1, t[h(t)]^{\gamma}|x|^{-\gamma-\alpha} q(|x|)\phi(|x|/4) + g\left(\frac{|x|}{2h(t)}\right)\right\}, \\
&\le c[h(t)]^{-d} \left(\min\left\{1, t[h(t)]^{\gamma}|x|^{-\gamma-\alpha} q(|x|)\phi(|x|/4)\right\} + g\left(\frac{|x|}{2h(t)}\right)\right).
\end{aligned}
$$

For $t > 1$ we have

$$
p_t(x+tb_{t^{1/\beta}}) \le ct^{-d/\beta}\left(\min\left\{1, t^{1+\frac{\gamma}{\beta}}|x|^{-\gamma-\alpha} q(|x|)\phi(|x|/4)\right\} + g\left(\frac{|x|}{2t^{1/\beta}}\right)\right), \quad x \in \mathbb{R}^d.
$$

It follows by [31, Lemma 25.5] that for every nonnegative function $f : \mathbb{R} \mapsto \mathbb{R}$ which is locally bounded and submultiplicative, i.e. $f(s_1+s_2) \le cf(s_1)f(s_2)$, there exist positive constants c_1, c_2 such that $f(s) \le c_1 \mathrm{e}^{c_2 s}$.

We observe that the function $1/\phi$ is submultiplicative by (4) and locally bounded therefore there exist c_3, c_4 such that $\phi(s) \ge c_3 \mathrm{e}^{-c_4 s}$ and this and (10) yield

$$
g(s) \le cs^{-\gamma-\alpha} q(s)\phi(s), \quad s > 0.
$$

For $t \in (0,1)$ we get

$$
p_t(x+tb_{t^{1/\alpha}}) \le ct^{-d/\alpha}\min\left\{1, t^{1+\frac{\gamma}{\alpha}}|x|^{-\gamma-\alpha} q(|x|)\phi(|x|/4)\right\}, \quad x \in \mathbb{R}^d. \quad \square
$$

6. Examples

6.1. Stable processes

We call a measure λ *degenerate* if there is a proper linear subspace M of $\mathbb{R}^d$ such that $\mathrm{supp}(\lambda) \subset M$; otherwise we call λ *nondegenerate*.

Let μ be a nondegenerate bounded measure on $\mathbb{S}$, $\alpha \in (0,2)$, and

$$
\nu(A) = \int_0^\infty \int_{\mathbb{S}} \mathbf{1}_A(s\theta)s^{-1-\alpha}\,\mathrm{d}s\,\mu(\mathrm{d}\theta), \quad A \subset \mathbb{R}^d. \tag{28}
$$

Let

$$
b = \begin{cases} \displaystyle\int_{|y|<1} y\nu(\mathrm{d}y) & \text{if } \alpha < 1, \\ 0 & \text{if } \alpha = 1, \\ \displaystyle -\int_{1<|y|} y\nu(\mathrm{d}y) & \text{if } \alpha > 1. \end{cases}
$$

If $\alpha = 1$ then we assume additionally that

$$
\int_{\mathbb{S}} \theta\mu(\mathrm{d}\theta) = 0.
$$

The corresponding semigroup $\{P_t : t \geq 0\}$ is called the α-stable semigroup and the stochastic process $\{X_t : t \geq 0\}$ is the α-stable Lévy process. We have (see, e.g., [31, Theorem 14.10])

$$\Phi(\xi) = a_\alpha \int_{\mathbb{S}} |\xi \cdot \theta|^\alpha \left(1 - \mathrm{i} \tan \frac{\pi\alpha}{2} \operatorname{sgn}(\xi \cdot \theta)\right) \mu(\mathrm{d}\theta), \quad \text{if } \alpha \neq 1, \tag{29}$$

and

$$\Phi(\xi) = a_1 \int_{\mathbb{S}} |\xi \cdot \theta| \left(1 + \mathrm{i}\frac{2}{\pi} \operatorname{sgn}(\xi \cdot \theta) \log |\xi \cdot \theta|\right) \mu(\mathrm{d}\theta), \quad \text{if } \alpha = 1, \tag{30}$$

where $a_\alpha = \frac{\pi}{2 \sin \frac{\pi\alpha}{2} \Gamma(1+\alpha)}$. From the fact that μ is nondegenerate it follows easily that $\Re(\Phi(\xi)) \geq c|\xi|^\alpha$. Of course $\Phi(a\xi) = a^\alpha \Phi(\xi)$ for every $a > 0$ and the semigroup has a scaling property, i.e.,

$$P_t(A) = P_1(t^{-1/\alpha} A), \quad A \subset \mathbb{R}^d,$$

and

$$p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x), \quad t > 0,\ x \in \mathbb{R}^d.$$

Using Theorem 1 for $t = 1$ with $q(s) \equiv \phi(s) \equiv 1$, and $\beta = \alpha$ we obtain

Corollary 9. *If ν is given by (28), μ satisfies (7) for some $\gamma \in [1, d]$, and $\{P_t : t \geq 0\}$ is the stable semigroup determinated by the characteristic exponent Φ given in (29) and (30) then we have*

$$p_t(x) \leq ct^{-d/\alpha} \min\{1, t^{1+\frac{\gamma}{\alpha}} |x|^{-\gamma-\alpha}\}, \quad x \in \mathbb{R}^d,\ t > 0.$$

We note that such estimates were first obtained in [4] for symmetric stable semigroups, and in [36] for non-symmetric stable semigroups. Our methods are different than those of [36].

6.2. Layered stable processes

Let μ be a nondegenerate bounded measure on $\mathbb{S}$, and

$$\nu(A) = \int_0^\infty \int_{\mathbb{S}} \mathbf{1}_A(s\theta) Q(\theta, s)\, \mathrm{d}s \mu(\mathrm{d}\theta), \quad A \subset \mathbb{R}^d, \tag{31}$$

where

$$Q(\theta, s) \approx s^{-\alpha-1} \mathbf{1}_{(0,1]}(s) + s^{-m-1} \mathbf{1}_{(1,\infty)}(s),$$

and $m > 2$. Let $b \in \mathbb{R}^d$ and

$$\Phi(\xi) = -\int \left(\mathrm{e}^{\mathrm{i}\xi \cdot y} - 1 - \mathrm{i}\xi \cdot y \mathbf{1}_{B(0,1)}(y)\right) \nu(\mathrm{d}y) - \mathrm{i}\xi \cdot b \tag{32}$$

A semigroup with the characteristic exponent (32) is called *layered stable semigroup* and was introduced by Houdré and Kawai in [19].

Corollary 10. *If ν is given by (31), μ satisfies (7) for some $\gamma \in [1, d]$, and $\{P_t : t \geq 0\}$ is the layered stable semigroup determinated by the characteristic exponent Φ given in (32) then the density p_t of P_t satisfies*

$$p_t(x + tb_{t^{1/\alpha}}) \leq ct^{-d/\alpha} \min\left\{1, t^{1+\frac{\gamma}{\alpha}} |x|^{-\gamma-\alpha} (1 + |x|)^{\alpha-m}\right\},$$

for $t \in (0, 1]$, $x \in \mathbb{R}^d$, *and*

$$p_t(x + tb_{\sqrt{t}}) \le c_1 t^{-d/2} \left(\min\left\{1, t^{1+\frac{\gamma}{2}} |x|^{-\gamma-\alpha}(1+|x|)^{\alpha-m}\right\} + \mathrm{e}^{\frac{-c_2|x|}{\sqrt{t}} \log\left(1+\frac{c_3|x|}{\sqrt{t}}\right)} \right),$$

for $t > 1, x \in \mathbb{R}^d$.

Proof. It follows from Lemma 10 in [34] that in this case

$$\Re\,(\Phi(\xi)) \approx |\xi|^2 \wedge |\xi|^\alpha, \quad \xi \in \mathbb{R}^d.$$

We let $q(s) = (1+s)^{\alpha-m}$, $\phi \equiv 1$, and $\beta = 2$ in Theorem 1. Of course

$$\int_0^r s^{1-\alpha}(1+s)^{\alpha-m}\,\mathrm{d}s \le \int_0^\infty s^{1-\alpha}(1+s)^{\alpha-m}\,\mathrm{d}s < \infty,$$

and (5) is also satisfied with $\beta = 2$ and the corollary follows from Theorem 1. □

6.3. Tempered stable processes

Let μ be a as above nondegenerate bounded measure on $\mathbb{S}$, and

$$\nu(A) = \int_0^\infty \int_{\mathbb{S}} \mathbf{1}_A(s\theta)s^{-1-\alpha} Q(\theta, s)\,\mathrm{d}s\mu(\mathrm{d}\theta), \quad A \subset \mathbb{R}^d. \tag{33}$$

If $Q(\theta, \cdot)$ is completely monotone for every $\theta \in \mathbb{S}$ then a process $\{X_t : t \ge 0\}$ which has the Lévy measure ν is called *tempered stable process*. Basic properties of tempered stable processes were investigated in [29]. We will focus on functions Q which decay exponentially: we will assume that

$$Q(\theta, s) \approx \mathrm{e}^{-cs},$$

for some constant $c > 0$. Let $b \in \mathbb{R}^d$, and

$$\Phi(\xi) = -\int \left(\mathrm{e}^{\mathrm{i}\xi\cdot y} - 1 - \mathrm{i}\xi \cdot y\mathbf{1}_{B(0,1)}(y)\right) \nu(\mathrm{d}y) - \mathrm{i}\xi \cdot b. \tag{34}$$

Taking $q \equiv 1$ and $\phi(s) = \mathrm{e}^{-cs}$ in Theorem 1 we obtain the following estimates.

Corollary 11. *If ν is given by (33), μ satisfies (7) for some $\gamma \in [1, d]$, and $\{P_t : t \ge 0\}$ is the tempered stable semigroup determinated by the characteristic exponent Φ given by (34), then the density p_t of P_t satisfies*

$$p_t(x + tb_{t^{1/\alpha}}) \le c_1 t^{-d/\alpha} \min\left\{1, t^{1+\frac{\gamma}{\alpha}} |x|^{-\gamma-\alpha}\mathrm{e}^{-c_2|x|}\right\},$$

for $t \in (0, 1]$, $x \in \mathbb{R}^d$, *and*

$$p_t(x + tb_{\sqrt{t}}) \le c_3 t^{-d/2} \left(\min\left\{1, t^{1+\frac{\gamma}{2}} |x|^{-\gamma-\alpha}\mathrm{e}^{-c_4|x|}\right\} + \mathrm{e}^{\frac{-c_5|x|}{\sqrt{t}} \log\left(1+\frac{c_6|x|}{\sqrt{t}}\right)} \right),$$

for $t > 1, x \in \mathbb{R}^d$.

6.4. Absolutely continuous and symmetric Lévy measure

Let ν be a symmetric measure, i.e., $\nu(A) = \nu(-A)$, and

$$\nu(\mathrm{d}x) \approx |x|^{-d-\alpha} q(|x|)\phi(|x|)\,\mathrm{d}x, \quad x \in \mathbb{R}^d \setminus \{0\}, \tag{35}$$

where q and ϕ satisfy (3)–(5) with $\beta = 2$. It follows from Theorem 1 here and Theorems 2 and 4 in [34] that the density p_t in this case satisfies the following estimates

$$\begin{aligned} c_1 \min\left\{t^{-d/\alpha}, t|x|^{-d-\alpha} q(|x|)\phi\,(2|x|)\right\} &\le p_t(x) \\ &\le c_2 \min\left\{t^{-d/\alpha}, t|x|^{-d-\alpha} q(|x|)\phi\,(|x|/4)\right\}, \end{aligned}$$

for $t \in (0, 1]$, $x \in \mathbb{R}^d$ and

$$\begin{aligned} &c_3 \min\left\{t^{-d/2}, t|x|^{-d-\alpha} q(|x|)\phi\,(2|x|)\right\} \\ &\quad \le p_t(x) \le c_4 \left(\min\left\{t^{-d/2}, t|x|^{-d-\alpha} q(|x|)\phi\,(|x|/4)\right\} + t^{-d/2} \mathrm{e}^{\frac{-c_5|x|}{\sqrt{t}} \log\left(1+\frac{c_6|x|}{\sqrt{t}}\right)}\right), \end{aligned}$$

for $t > 1$, $x \in \mathbb{R}^d$.

In particular for the relativistic α-stable Lévy process, which is investigated, e.g., in [30,24,15,5], we have

$$\begin{aligned} \nu(D) &= c_1 \int_D |y|^{-d-\alpha} K_{d,\alpha}(|y|)\,\mathrm{d}y \\ &= c_2 \int_{\mathbb{S}} \int_0^\infty \mathbf{1}_D(s\theta) s^{-1-\alpha} K_{d,\alpha}(s)\,\mathrm{d}s\sigma(\mathrm{d}\theta), \quad D \subset \mathbb{R}^d, \end{aligned}$$

where σ is the standard isotropic surface measure on $\mathbb{S}$ and

$$K_{d,\alpha}(s) = s^{d+\alpha} \int_0^\infty \mathrm{e}^{-u} \mathrm{e}^{-\frac{s^2}{4u}} u^{\frac{-2-d-\alpha}{2}}\,\mathrm{d}u, \quad s > 0.$$

We have (see [15])

$$K_{d,\alpha}(s) \approx (1+s)^{\frac{d+\alpha-1}{2}} \mathrm{e}^{-s}$$

and in this case we get

$$c_1 \min\left\{t^{-d/\alpha}, t|x|^{-d-\alpha} \mathrm{e}^{-2|x|}\right\} \le p_t(x) \le c_2 \min\left\{t^{-d/\alpha}, t|x|^{-d-\alpha} \mathrm{e}^{-|x|/5}\right\},$$

for $t \in (0, 1]$, $x \in \mathbb{R}^d$ and

$$\begin{aligned} c_3 \min\left\{t^{-d/2}, t|x|^{-d-\alpha} \mathrm{e}^{-2|x|}\right\} &\le p_t(x) \le c_4 \left(\min\left\{t^{-d/2}, t|x|^{-d-\alpha} \mathrm{e}^{-|x|/5}\right\}\right. \\ &\quad \left. +\; t^{-d/2} \mathrm{e}^{\frac{-c_5|x|}{\sqrt{t}} \log\left(1+\frac{c_6|x|}{\sqrt{t}}\right)}\right), \end{aligned}$$

for $t > 1$, $x \in \mathbb{R}^d$. The sharp estimates of the transition densities of the relativistic process for small t are given in [11].

Acknowledgement

The research was supported by grant MNiSW N N201 397137.

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